Exercise 18

A freshly brewed cup of coffee has temperature 95°C in a 20°C room. When its temperature is 70°C, it is cooling at a rate of 1°C per minute. When does this occur?

Solution

Assume that the rate of decrease of the corpse's temperature is proportional to the difference between the corpse's temperature and the surrounding temperature T_s.

$$\frac{dT}{dt} \propto -(T - T_s)$$

The minus sign is included so that when the surroundings are cooler (hotter) than the corpse, dT/dt is negative (positive). Change this proportionality to an equation by introducing a positive constant k.

$$\frac{dT}{dt} = -k(T - T_s)$$

To solve this differential equation for T, make the substitution $y = T - T_s$.

$$\frac{dT}{dt} = -ky$$

Differentiate both sides of the substitution with respect to t to write the derivative in terms of y: $\frac{dy}{dt} = \frac{d}{dt}(T - T_s) = \frac{dT}{dt}$.

$$\frac{dy}{dt} = -ky$$

Divide both sides by y.

$$\frac{1}{y}\frac{dy}{dt} = -k$$

Rewrite the left side by using the chain rule.

$$\frac{d}{dt}\ln y = -k$$

The function you take a derivative of to get $-k$ is $-kt + C$, where C is any constant.

$$\ln y = -kt + C$$

Exponentiate both sides to get y.

$$e^{\ln y} = e^{-kt+C}$$

$$y = e^C e^{-kt}$$

Use a new constant A for e^C.

$$y(t) = Ae^{-kt}$$

Now that the differential equation has been solved, change back to the original variable T, the corpse's temperature.

$$T - T_s = Ae^{-kt}$$

As a result,

$$T(t) = T_s + Ae^{-kt}.$$

Since the room's temperature is 20°C, $T_s = 20$.

$$T(t) = 20 + Ae^{-kt}$$

Use the fact that the cup's initial temperature is 95°C.

$$95 = 20 + Ae^{-k(0)} \quad \rightarrow \quad A = 95 - 20 = 75$$

Consequently,

$$T(t) = 20 + 75e^{-kt}.$$

Use the two given pieces of information to construct a system of equations for k and t, that is, that when the rate of change of the temperature is -1 the temperature is 70.

$$\begin{cases} T(t) = T_s + Ae^{-kt} \\ \dfrac{dT}{dt} = -k(T - T_s) \end{cases} \Rightarrow \begin{cases} 70 = 20 + 75e^{-kt} \\ -1 = -k(70 - 20) \end{cases}$$

Solve the second equation for k.

$$1 = k(50)$$

$$k = \frac{1}{50} = 0.02$$

Then plug it into the first equation and solve for t.

$$70 = 20 + 75e^{-(0.02)t}$$

$$50 = 75e^{-0.02t}$$

$$\frac{50}{75} = e^{-0.02t}$$

$$\ln\frac{50}{75} = \ln e^{-0.02t}$$

$$\ln\frac{2}{3} = (-0.02t)\ln e$$

$$t = -\frac{\ln\frac{2}{3}}{0.02} \approx 20.2733 \text{ minutes}$$

Find out how many seconds 0.2733 minutes is.

$$0.2733 \text{ minutes} \times \frac{60 \text{ seconds}}{1 \text{ minute}} \approx 16.398 \text{ seconds}$$

Therefore, it takes about 20 minutes and 16 seconds for the coffee's temperature to go from 95°C to 70°C in a 20°C room.